JOURNAL OF COMPUTATIONAL PHYSICS **142,** 611–614 (1998)
ARTICLE NO. CP985950

NOTE

On a Mistaken Notion of "Proper Upwinding"

In computational fluid dynamics the Riemann problem, an initial value problem with piecewise constant initial data having only one discontinuity, plays a special role. An exact or an approximate solution to this problem, whose relevance for numerical computations was originally suggested in [3], is used in many modern codes for hyperbolic conservation laws to find the fluxes of conserved quantities, such as mass, momentum, and energy, across the interfaces of computational cells. Deriving the flux in such a way allows many properties of the governing equations to be directly built into the numerical method, in particular properties having to do with the directionality of wave propagation. For example, if all information propagates through an interface in only one direction, then one would want to evaluate the interface flux based only on the upstream state, so that the flux is wholly upwind. How can such a case be detected? One situation appears to be obvious. Suppose both initial states in a Riemann problem are supersonic in the same direction. Then, with all the information initially propagating in this direction, it seems natural to assume that it will continue to propagate in one direction even after the two states interact. In fact, this is exactly what many numerical schemes will do. In particular, it is done by any code that determines the domain of influence before solving the Riemann problem. Some writers even take this property to be a necessary condition for a flux function to be "properly upwind." In this note we want to point out that this notion of "proper upwinding" is erroneous.

From the theory of hyperbolic conservation laws it is well known (see, for example, [4]) that the speed of an entropy-satisfying "convex" wave lies in the range determined by the characteristics of an appropriate family immediately to the left and to the right of the wave. In the scalar case, which admits waves of only one family, this means that in a Riemann problem the characteristics computed from the initial data *a priori* determine the domain of influence to which the resulting wave will be confined. Thus, if the characteristics on both sides of the initial discontinuity are in the same direction, the interface flux can always be upwinded. This result is so natural that it is tempting to extend it to the non-scalar case as in Fig. 1. However, unless the initial states in a Riemann problem can be connected by a single wave, they are not neighboring states for $t > 0$. The possibility is therefore open that the domain of influence of the initial discontinuity extends outside the union of the domains of dependence defined by the two initial states (see Fig. 1).

That this can indeed happen is easy to demonstrate for gas dynamics. Consider the following Riemann problem. Let the initial left state L be supersonic in the positive direction (see Fig. 2). From the $u - p$ diagram (see [1]) it is clear that for any such left state we can

0021-9991/98 $25.00

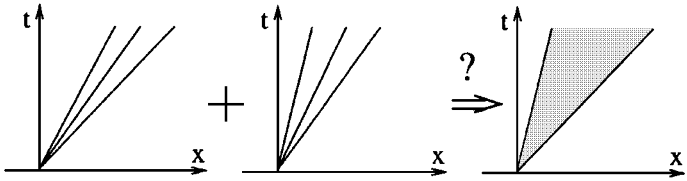

FIG. 1. Is the domain of influence in a Riemann problem bounded by the characteristics of the initial states? Not necessarily.

always find a right state R, also with positive velocity, whose pressure is high enough to produce a left going contact discontinuity. We can ensure that the state R is also supersonic simply by making it sufficiently dense. Then by construction the contact discontinuity is left-running, and *a fortiori* the $(u - a)$ wave also. For example, it is easy to check that for $\gamma = 1.4$, $u_R = u_L$, $a_R = a_L$, $M_R = M_L = 1.1$, and $p_R = 100 p_L$ the resulting velocity of the contact discontinuity is $u_I \approx -0.52 a_L$, where γ is the polytropic constant, u is the velocity, a is the sound speed, M is the Mach number, and p is the pressure of the gas.

Thus we see that so-called "proper upwinding" is not a feature of the exact Riemann solution and therefore cannot be a necessary feature of an approximate one. Nevertheless, many authors have assumed that it should be. Vinokur [6] noted that Roe's scheme [5] could predict eigenvalues outside the range spanned by the initial data and assumed this to be a defect of the scheme. This criticism was accepted in [2]. However, a careful analysis of the first two cases treated in the Appendix of [6] reveals that their exact solutions feature weak left-running shocks. Thus Roe's scheme in these cases approximates the correct physical behavior. Honesty, however, compels the second author to admit that this feature was a matter of serendipity rather than conscious design.

A practical motivation for accepting the upwind flux under some circumstances is that it could speed up the calculations, and we now derive a sufficient condition. Let both the left and the right states in a Riemann problem be supersonic in the positive direction. Obviously, upwinding will be possible if no left-going shock wave appears. Note that this allows an expansion wave or a right-going shock wave bounding the left state. Since the expansion wave case always results in upwinding, we will concentrate on the shock wave case. If the left state undergoes a shock wave transition, the velocity of the resulting shock wave is

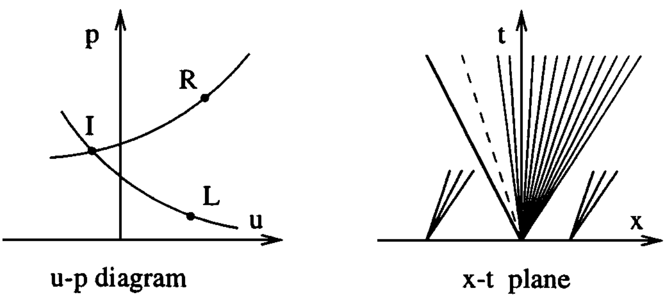

FIG. 2. A Riemann problem in which the direction of the intermediate state velocity is opposite to the velocities of the initial supersonic states.

given by

$$v_s = u_L - a_L \sqrt{1 + \frac{\gamma + 1}{2\gamma}\left(\frac{p_*}{p_L} - 1\right)}, \tag{1}$$

where p_* is the pressure of the adjacent intermediate state (see [1] for this and other formulae used below). The shock velocity is positive if

$$\frac{p_*}{p_L} < 1 + \frac{2\gamma}{\gamma + 1}\left(M_L^2 - 1\right). \tag{2}$$

Then the corresponding intermediate state velocity, u_*, given by

$$u_* = u_L - \frac{a_L}{\gamma} \frac{p_*/p_L - 1}{\sqrt{1 + ((\gamma + 1)/2\gamma)(p_*/p_L - 1)}} \tag{3}$$

has to satisfy

$$\frac{u_*}{a_L} > \frac{(\gamma - 1)M_L^2 + 2}{(\gamma + 1)M_L}. \tag{4}$$

The intermediate velocity can alternatively be expressed in terms of the right state parameters. The right state may undergo either an expansion or a shock wave transition. In the first case

$$u_* = u_R + \frac{2}{\gamma - 1} a_R \left(\left(\frac{p_*}{p_R}\right)^{\frac{\gamma - 1}{2\gamma}} - 1\right), \tag{5}$$

while in the second one

$$u_* = u_R + \frac{a_R}{\gamma} \frac{p_*/p_R - 1}{\sqrt{1 + ((\gamma + 1)/2\gamma)(p_*/p_R - 1)}}. \tag{6}$$

We could check whether the right state shocks or expands, however, it can be shown that

$$\frac{2}{\gamma - 1}\left(\left(\frac{p_*}{p_R}\right)^{\frac{\gamma - 1}{2\gamma}} - 1\right) \leq \frac{1}{\gamma} \frac{p_*/p_R - 1}{\sqrt{1 + ((\gamma + 1)/2\gamma)(p_*/p_R - 1)}} \tag{7}$$

for all meaningful values of γ. Therefore a lower bound on the intermediate velocity can always be found using the expansion curve. This simplification makes physical sense since the right state, whose pressure must be high to produce a left-running shock wave, is most likely to expand.

Using Eqs. (2), (4), and (5) we can find that

$$\frac{(\gamma - 1)M_L^2 + 2}{(\gamma + 1)M_L} < \frac{u_R}{a_L} + \frac{2}{\gamma - 1}\frac{a_R}{a_L}\left(\left(\frac{p_*}{p_L}\right)^{\frac{\gamma - 1}{2\gamma}}\left(\frac{p_L}{p_R}\right)^{\frac{\gamma - 1}{2\gamma}} - 1\right) \tag{8}$$

$$< \frac{u_R}{a_L} + \frac{2}{\gamma - 1}\frac{a_R}{a_L}\left(\left(\frac{p_L}{p_R}\right)^{\frac{\gamma - 1}{2\gamma}}\left(1 + \frac{2\gamma}{\gamma + 1}\left(M_L^2 - 1\right)\right)^{\frac{\gamma - 1}{2\gamma}} - 1\right). \tag{9}$$

Combined with $u_L > a_L$ and $u_R > a_R$ the above condition yields the following sufficient criterion for upwinding in the positive direction,

$$\frac{(\gamma-1)M_L^2+2}{(\gamma+1)M_L} < \frac{a_R}{a_L}\left[M_R + \frac{2}{\gamma-1}\left(\left(\frac{p_L}{p_R}\right)^{\frac{\gamma-1}{2\gamma}}\left(1+\frac{2\gamma}{\gamma+1}\left(M_L^2-1\right)\right)^{\frac{\gamma-1}{2\gamma}} - 1\right)\right]. \tag{10}$$

A similar expression can be easily obtained for upwinding in the negative direction when $u_L < -a_L$ and $u_R < -a_R$. However, Eq. (10) is fairly complicated and expensive to compute, therefore it is not clear if the obtained criterion is useful for practical calculations.

Our objective in writing this note is chiefly to correct a theoretical misconception. In order to see whether the misconception has serious practical consequences for unsteady calculations we have constructed various data with two supersonic initial states that should generate a left-going shock. Numerical results were then obtained using several flux functions. One might think that "properly upwinded" fluxes would be unable to initially predict the correct flow. Indeed, at the first timestep, no left-going disturbance can be generated. At later timesteps, however, Riemann problems are solved involving "intermediate" states, and in all cases the correct wave pattern emerges quite soon; not a surprising result for a convergent scheme. Thus, in unsteady calculations no serious problems occur. In steady-state calculations, however, the situation may potentially be different. It is likely that an optimized implementation of a particular flux function utilizes "proper upwinding" which involves a simple logical switch. Because of this switch, some flux functions, in particular Roe's flux function, lose Lipshitz continuity, which in turn may interfere with convergence. We are not currently aware whether such an effect has ever been observed. Nevertheless, we have shown that the notion of "proper upwinding" as sometimes proposed is in fact mistaken. It is apparently just good fortune if it causes no harm.

REFERENCES

1. R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience, New York, 1948).
2. B. Einfeldt, C. D. Munz, P. L. Roe, and B. Sjögreen, On Godunov-type methods near low densities, *J. Comput. Phys*. **92**, 273 (1991).
3. S. K. Godunov, A difference scheme for numerical computation of discontinuous solutions of hydrodynamic equations, *Mat. Sb*. **47**(3), 271 (1959). [In Russian]
4. P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves* (SIAM, Philadelphia, 1973).
5. P. L. Roe, Approximate Riemann solvers, parameter vectors, and difference schemes, *J. Comput. Phys.* **43**, 357 (1981).
6. M. Vinokur, An analysis of finite-difference and finite-volume formulation of conservation laws, *J. Comput. Phys.* **81**, 1 (1989).

Received April 7, 1997; revised February 17, 1998

Timur Linde
Philip Roe
W. M. Keck Foundation Laboratory for Computational Fluid Dynamics
Aerospace Engineering, The University of Michigan
Ann Arbor, Michigan 48109-2118
E-mail: linde@engin.umich.edu and philroe@engin.umich.edu